Mathematical Foundations of Complex Networked Information Systems

CIME school/ Newcom++ school

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ABSTRACT:

In the past twenty years we have being witnessing a revolution in information and communication technologies, consisting in an astonishing progress in many different research fields. Consider for instance the innovations not only in the classical fields of wireless communication (design of high performance codes very close to the Shannon limit), computer engineering (parallel computing, computer vision, data mining), bioinformatics (genoma interpretation) and VLSI technologies, but also in new fields which will have a great impact in the future such as wireless sensor networks and smart dust, air traffic control, large scale surveillance systems, unmanned mobile multi-vehicle systems, biological networks interpretation and so on.

All of these problems have a common feature, namely they typically deal with complex systems and aste for the solution of large scale optimization problems requiring a prohibitive amount of computation effort even in low dimension. Moreover, the complexity of such systems requires solutions which need to share robustness and adaptation. In spite of the inherent difficulty of these issues, fundamental contributions have already appeared towards the solution of such problems. All these contributions are based on the exploitation of the common structure that, in many cases, these complex systems exhibit: they are often the result of the interactions of numerous but rather simple entities. The complexity is then a consequence of the interactions architecture described by a network and the solutions can be based on mathematical models which allow to predict how the designed local rules give rise to a global behavior obeying prescribed specifications.

The mathematical tools which are playing a fundamental role in addressing these new scientific issues come mainly from combinatorics, probability theory, and statistical mechanics. The concept of graph plays a prominent role. Besides being the basic model of any communication network, it also appears as a natural model to describe interactions among variables: an instance of this are graphical models in modern coding theory.

Random graphs are, in particular, the right model to describe complex networks (e.g. internet, wireless communication networks, sensor networks). Randomness enter at different levels: as a smart technique to mimic the development of complex networks, as an unavoidable modelling technique of faults or noise in the communication links, or also as a useful technique to find typical codes or algorithms with good properties. For a deep analysis of these random models certain probabilistic

techniques play a fundamental role. Among these we can list concentration inequalities, large deviation theory and percolation. In the situation in which we have an aggregation of simple entities and in which the complexity rises from the network of interactions, statistical mechanics becomes a natural approach.

This school will propose an introduction to some of the fundamental scientific issues emerging in these disciplines. Research in this field necessarily has to cope with a double difficulty: the problem of constructing coherent mathematical models sufficiently rich to be able to describe the 'real' complex networks we want to study and, on the other side, simple enough to be amenable for a general mathematical theory. While elegant and complete mathematical results already exist for certain specific problems, it is true that the mathematical theory of complex networks is far from being complete and many fundamental open problems still wait for an answer.

Our school consists of five courses. There will be a course on random graphs (by Prof. Bollobas) which will provide all the probabilistic tools needed to deal with these fundamental objects and will propose a number of different ways to model 'randomness'. It will provide the mathematical tools to be able to rigorously study networks in a variety of different scientific and technological contexts.

One course will be devoted to communication networks, in particular to the issues of the transmission of information along a network where there are simultaneously many potential receivers and many potential transmitters. These issues are clearly of fundamental technological importance and, at the same time, they constitute a source of challenging mathematical problems. The course delivered by Prof. Kumar will study the basic information theoretic aspects of wireless communication networks.

Other two courses will instead focus on the analysis of distributed algorithms over networks and, more generally, graphical models. Particular emphasis will be on the famous message passing algorithms which have applications in a broad variety of contexts including artificial intelligence, distributed inferential statistics, coding theory, and combinatorial optimization. The course by Prof. Zecchina will focus on the statistical physics interpretation of these algorithms with special attention to applications in classical combinatorial optimization problems. The course by Prof. Wainwright will instead focus on the probabilistic aspects of message passing algorithms and their relation with the relaxation techniques in optimization theory.

SPEAKERS:

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Random graphs

About fifty years ago, three closely related subjects emerged independently of each other. Broadbent and Hammersley founded the theory of percolation, the study of random subgraphs of lattice-like graphs, inspired by the way a liquid seeps through a porous medium. Gilbert introduced random geometric graphs aimed at modelling wireless communication in the plane or space. Finally, Erd?os and Rényi founded the theory of random graphs, the study of random subgraphs of complete graphs, in order to prove the existence of paradoxical graphs, and to find out what a typical graph with n vertices and m = m(n) edges looks like, as n tends to infinity. Simultaneously with these developments, statistical physicists carried out many computer experiments and did much non-rigorous work not only on percolation, but also on general disordered systems like the Ising model and the Potts model. With the exception of the second, the theory of geometric random graphs, these have become huge areas. For many years, these theories developed independently of each other, but in the past decade or so much cross-fertilization has been going on: methods of one field are applied to problems in another so that, by now, many mathematicians consider these areas as different aspects of the same big area, the study of large-scale random combinatorial systems. In spite of the tremendous amount of research mathematicians and physicists have done in these fields, there are many more important open problems than results. Not surprisingly, the ErdösRényi theory of random graphs is the richest in sharp mathematical results, after all, the ER random graph model is just the so-called mean-field percolation model. Nevertheless, the ER theory is extremely important because its results can be viewed as examples of the kind of precise results we hope to obtain in the other fields, and because the tools we develop are easiest tested in this theory. Needless to say, sixty lectures would not suffice to give a thorough grounding in these theories, let alone six; still, in our six lectures we hope to give a taste of the results in two of these fields: the Erdös Rényi theory of random graphs and the theory of geometric random graphs. Here is a tentative plan of the lectures.

(i) Probabilistic tools and the classical models. We shall introduce some of the tools used to study random graphs, concentrating on correlation and isoperimetric inequalities. Turning to random graphs, we shall define some of the basic models, and prove their basic properties.

(ii) Sharp thresholds and properties of random graphs. One of the surprising phenomena discovered by Erdös and Rényi is the sudden emergence of various monotone properties. Given a monotone property of graphs, there is some function $m^*(n)$ such that a typical graph with n vertices and a little fewer than $m^*(n)$ edges does not have this property, but one with a little more than $m^*(n)$ edges does. In the past few years rather sophisticated results have been proved about the sharp thresholds or sharp phase transitions: we shall present some of these results and apply them to various properties.

(iii) Random geometric graphs: the Gilbert disc model and ad hoc sensor networks. For us, a geometric graph will be one whose vertices are in the plane or space, or in some metric space. For example, a planar network of sensors is modelled by a geometric graph in which two vertices joined by an edge if their sensors can communicate with each other. Choosing the points at random with some probability distribution, we have a random geometric graph. For example, in the Gilbert disc model two points are joined if their distance is no more than some parameter r. In the lectures we shall prove some recent results about these networks.

Prof. Riccardo Zecchina
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Statistical physics of constraint satisfaction networks

Table of contents:

(i) Introduction to statistical physics: Combinatorial interpretation. Phase transition in the Ising problem. Phase transitions in spin glasses

(ii) Statistical Physics of random Constraint Satisfaction Problems Random k-xor-sat phase diagram and its rigorous derivation Statistical mechanics of random k-sat and coloring. The phase diagram from the cavity method. Clustering of solutions Rigorous results on clustering

(iii) Message-passing algorithms. From the cavity method to Survey Propagation (SP)algorithms. Some applications to coding, networks and computational biology.

(iv) Perspectives and open problems

Prof. P.R. Kumar
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Mathematical foundations of wireless networks

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(i) Capacity of Deterministic Wireless Networks We consider a model of a wireless network with n nodes, that treats interference as noise. We show that the maximum order of the transport capacity, a measure of how much information a network can pump, is $O(1/\sqrt{n})$. This is the maximum over the locations of the nodes, the origin-destinations pairs, and traffic demands, when the network is optimally operated.

(ii) Capacity of random wireless networks We next turn to the case where the locations of the nodes are random, and the origin-destination pairs are also chosen randomly. We prove a sharp cutoff result. We show that under optimal operation there are two constants $0 < c_1 < c_2$ such that:

$$\lim_{n\to\infty} \Pr(\text{A throughput of } c_1/\sqrt{n\log n} \text{ can be provided to each of the } n \text{ source destin. pairs}) = 1$$

and

$$\lim_{n\to\infty} \Pr(\text{A throughput of } c_2/\sqrt{n\log n} \text{ can be provided to each of the n source destin. pairs}) = 0$$

(iii) A brief introduction to classical information theory. We prove the asymptotic equipartition property, and then establish the properties of typical sets. Then we prove the now classical result of Shannon that the capacity of discrete memoryless channel is $\max_p(x)I(X;Y)$, where $I(X;Y)$ denotes the mutual information between the sender and the receiver.

(iv) Network information theory Next we examine the case of networks. We provide an account of the results for the multiple access channel, the broadcast channel, and the Slepian-Wolf Theorem. Then we provide an account of information-theoretic scaling laws for wireless networks.

(v) Function computation in wireless networks We address the problem of how functions of information present at the nodes of a wireless network can be computed. We study the class of type-threshold and type-sensitive functions, and exhibit scaling laws for the rates at which they can be computed.

(vi) Synchronizing clocks in wireless networks. In wireless networks that interact with the physical environment, it is important for all the nodes in the network to synchronize their clocks. We begin by characterizing what are the invariants that cannot be determined concerning clocks in wireless networks, no matter how many time-stamped messages are exchanged between nodes. Then we turn to algorithms for spatially smoothing errors, and present an analysis of the asymptotic variance as a function of network size. Finally we show that in random two-dimensional networks the error is $O(1)$. We also address the rate of convergence in terms of the degrees of nodes and the edge connectivity index.

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Graphical models and distributed algorithms: Message-passing and relaxations

Graphical models provide a flexible framework for capturing dependencies among large collections of random variables, and are by now an essential component of the statistical machine learning toolbox. Any application of graphical models involves a core set of computational challenges, centered around the problems of marginalization, mode-finding, parameter estimation, and structure estimation. Although efficiently solvable for graphs without cycles (trees) and graphs of low treewidth more generally, exact solutions to these core problems are computationally challenging for general graphical models with large numbers of nodes and/or state space sizes. Consequently, many applications of graphical models require efficient methods for computing approximate solutions to these core problems. The past decade and a half has witnessed an explosion of activity on approximate algorithms for graphical models. This tutorial will show how a wide class of methods—-including mean field theory, sum-product or belief propagation algorithms, expectation-propagation, and max-product algorithms—-are all variational methods, meaning that they can be understood as algorithms for solving particular optimization problems on graphs. The perspective also forges connections to convex optimization, including linear programming and other type of conic relaxations.

Table of contents:

(i) Basics of graphical models. Directed models, undirected models, factor graphs. Marginalization, MAP problem, parameter/structure learning. Junction tree formalism

(ii) Exponential families and maximum entropy. Maximum entropy problem. Conjugate duality. Mean parameters and global validity.

(iii) Bethe-Kikuchi, sum-product, and expectation-propagation. Bethe approximation and belief propagation. Kikuchi clustering and generalization. Expectation-propagation

(iv) Mean field methods and learning. Naive and structured mean field. Variational EM and Bayes

(v) LP relaxations and max-product. Basic tree-based LP relaxation. Reweighted max-product and LP duality. Higher-order LP relaxations.

(vi) Other convex relaxations. Second-order cone programming and relation to LPs. Semidefinite relaxations.